Quadratic Dynamic and Excess Intersection

X smooth variety $/k$

$$CH^n(X) = \bigoplus_{x \in X^{(n)}} \mathbb{Z} \Big/ \begin{matrix}\text{rational}\\ \text{equivalence}\end{matrix}$$

Alternative definition (Bloch - Kato) :

$$CH^n(X) := H^n(X, K_*^M)$$

<u>oriented Chow ring</u>: $L \to X$ line bundle

$$\widetilde{CH}^n(X,L) := H^n(X, K_*^{MW}(L))$$

Rost-Schmid complex:

$$0 \to \bigoplus_{x \in X^{(0)}} K_n^{MW}(k(x), L) \to \dots \to \bigoplus_{x \in X^{(n)}} K_0^{MW}\left(k(x), L \otimes \det^{-1} \frac{m_x}{m_x^2}\right)$$

$$\to \dots$$

F field eg $F = k(x)$

$K_0^{MW}(F) \cong GW(F) =$ group completion (non-deg symm bilinear form / F)

GW(F) is generated by

$\langle u \rangle =$ class of $F \times F \to F$
$(x, y) \mapsto uxy$

$u \in F^\times / (F^\times)^2$

relations: 1) $\langle u \rangle = \langle u v^2 \rangle$

2) $\langle u v \rangle = \langle u \rangle \langle v \rangle$

3) $\langle u \rangle + \langle v \rangle = \langle u v \rangle + \langle u v (u+v) \rangle$

[4) $\langle u \rangle + \langle -u \rangle = \langle 1 \rangle + \langle -1 \rangle$]

$$\begin{matrix} \langle 1 \rangle + \langle -1 \rangle \\ \| \\ \mathbb{H} \end{matrix}$$

$\mathbb{H}$ = hyperbolic form

"So elements in $\widetilde{CH}^n(X, L)$ are quadratic forms over $k(x)$ $x \in X^{(n)}$

twisted by some line bundle".

$$\begin{aligned} F^\times &\longrightarrow K_*^{MW}(F) \\ u &\longmapsto \langle u \rangle \end{aligned}$$

$$K_*^{MW}(F, L) := K_*^{MW}(F) \otimes_{\mathbb{Z}[F^\times]} \mathbb{Z}[L^\times]$$

L = 1-dim F-VS

Properties:

1) $K^{MW}_*(L) \longrightarrow K^M_*$

$\rightsquigarrow \widetilde{CH}^n(X, L) \longrightarrow CH^n(X)$

2) $f : Y \to X$ $\quad L \to X$ line bundle

$f^* : \widetilde{CH}^n(X, L) \longrightarrow \widetilde{CH}^n(Y, f^*L)$

3) $f : Y \to X$ proper $\quad d = \dim X - \dim Y$

$f_* : \widetilde{CH}^n(Y, f^*L \otimes \omega_{Y/k}) \to \widetilde{CH}^{n+d}(X, L \otimes \omega_{X/k})$

Euler class

$$\begin{tikzcd}
V \arrow[r, "\pi"] & Y \arrow[l, bend left, "s_0"]
\end{tikzcd}$$
VB rk $V = r$

s_0 = zero section

$$\begin{tikzcd}
\widetilde{CH}^0(Y) \arrow[r, "s_{0*}"] & \widetilde{CH}^r(V, \pi^*\det^{-1} V) \arrow[r, "s_0^*"] & \widetilde{CH}^r(V, \det^{-1} V) \\
\langle 1 \rangle \arrow[u, phantom, sloped, "\in"] \arrow[rr, mapsto] & & e(V) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$= $ Euler class

$\omega_{Y/k} \otimes \omega_{Y/k}^{-1}$

$\pi^* \det^{-1} V \cong \omega_{V/k} \otimes \pi^* \omega_{Y/k}^{-1}$

$s_0^* \pi^* \omega_{Y/k}^{-1} = \omega_{Y/k}^{-1}$

Euler number

$\operatorname{rk} V = r = \dim Y$ $\qquad$ $p : Y \to \operatorname{Spec} k$ proper

V is relatively oriented

by $\rho := \omega_{Y/k} \otimes \det V \xrightarrow{\cong} L^{\otimes 2}$ $\qquad$ $L \to Y$ ine bundle

$$\begin{tikzcd}
\widetilde{CH}^r(Y, \det^{-1} V) \arrow[r, phantom, "\cong"] & \widetilde{CH}^r(Y, \cancel{L^{\otimes -2}} \otimes \omega_{Y/k}) \arrow[d] \\
e(V) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & n(V,\rho) \in \widetilde{CH}^0(\operatorname{Spec} k) = GW(k)
\end{tikzcd}$$

Euler number :=

Ex: $V = TY$

$$\begin{tikzcd}
\widetilde{CH}^r(Y, \underset{\substack{\shortparallel \\ \omega_{Y/k}}}{\det^{-1}TY}) \arrow[r, "p_*"] & GW(k) \\
e(V) \arrow[u, phantom, sloped, "\in"] \arrow[r, bend right] & \chi(Y) \\
& \text{Euler characteristic} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

(Levine)

Computation (Kass - Wickelgren)

$$\begin{tikzcd} V \arrow[r, "\pi"] & Y \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

$\mathrm{rk}\, V = r = \dim Y$

$p : Y \to \mathrm{Spec}\, k$ smooth + proper

$\rho : \omega_{Y/k} \otimes \det V \xrightarrow{\cong} L^{\otimes 2}$

Assume $\{\sigma = 0\} = Z = \{x_1, \dots, x_s\} \subset X$ closed pts

$k(x_i)/k$ separable

Then $n(V, \rho) = \sum_{x \in Z} \mathrm{ind}\, x$

Choose Nisnevich coordinates around x

$\psi : \underset{\overset{\cup}{x}}{U} \longrightarrow \mathbb{A}^r$ st ψ induces an iso on $k(x)$

+ $V|_U \cong U \times \mathbb{A}^r$ "compatible" with rel orientation ρ.

Then $g = (f_1, \dots, f_r) : \mathbb{A}^r \longrightarrow \mathbb{A}^r$

and $\operatorname{ind} x = \operatorname{Tr}_{k(x)/k} \langle \det \frac{\partial f_i}{\partial t_j}(x) \rangle$

$$V \times V \xrightarrow{b} k(x) \xrightarrow{\mathrm{Tr}_{k(x)/k}} k \qquad \text{form}/k$$

$$\rightsquigarrow \mathrm{Tr}_{k(x)/k}(b)$$

<u>Ex</u> (Kass - Wickelgren)

Lines on a smooth cubic surface

$$= n(\mathrm{Sym}^3 S^* \to Gr(2,4))$$

$$= 15\langle 1 \rangle + 12 \langle -1 \rangle \quad \in GW(k)$$

$$\downarrow \mathrm{rk}$$

$$27 = \text{classical count}$$

$GW(\mathbb{R})$ is generated by

$u \in \mathbb{R}^\times / (\mathbb{R}^\times)^2 = \{\pm 1\}$

1) what if $\det \frac{\partial f_i}{\partial t_j}(x) = 0$?

A: EKL-form (Eisenbud-Levine, Khimshiashvili / $\mathbb{R}$

Kass-Wickelgren,

Brazelton-Burklund-

McKean-Montoro-Opie)

2) What if $\{\sigma = 0\} = Z$
st Z has a positive dim
component?

Excess intersection formula

(Fasel, Déglise-Jin-Khan, Bachmann-Wickelgren)

$Z \hookrightarrow Y$ regularly embedded

then

$$n(V; \rho) = \sum_{\substack{Z' \subset Z \\ \text{clopen}}} n(\mathcal{E}|_{Z'}, \rho')$$

$$\mathcal{E} = \operatorname{coker}\left(C_z Y \hookrightarrow V|_z \right)$$

Quadratic dynamic intersection

View $\sigma_t = \sigma + t\sigma' + t^2\sigma'' + \ldots$

as a section $V_{k((t))} \to Y_{k((t))}$

$\rightsquigarrow n(V_{k((t))}) \in GW(k((t)))$

Euler classes / numbers commute with base change.

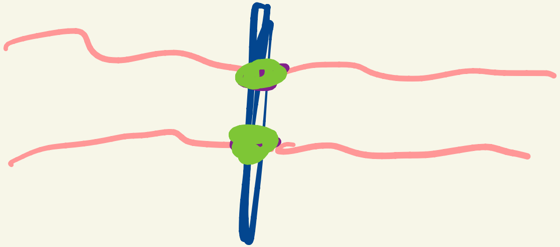
injective $\rightarrow$ $i : GW(k) \longrightarrow GW(k((t)))$

$$\langle u \rangle \longmapsto \langle u \rangle$$

$$n(V_k) \longmapsto n(V_{k((t))})$$

$Z = \{\sigma = 0\}$

$Z_t = \{\sigma_t = 0\}$



$Z_t \times_{k[[t]]} k((t)) = Z^*$

$Z' :=$ closure of Z^* in Z

Z'_0 $\qquad \mu_t : \widetilde{CH}(Z') \to \widetilde{CH}(Z'_0)$